COMMUNICATIONS TO THE EDITOR

Some Remarks on the Problem of Drainage of Fluids on Vertical Surfaces

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Prediction of the thickness of the liquid film during draining of the liquid on a vertical surface has considerable importance in various practical cases such as in lubrication, in drainage and in estimation of the amount of liquid adhering to walls of pipetes, burettes, and capillary viscometers after emptying.

The thickness of the liquid film draining from a vertical surface depends upon the interrelationship among surface tension, the gravitational force, and the viscous force. The effect of surface tension depends upon the curvature of the outer surface of the liquid layer which can be expressed as the change of the coating thickness h with height. For the case of a flat plate which is partially immersed in the liquid, the change of thickness with distance from the top will be relatively small at places well above the main surface of the liquid bath.

However, near the bath surface the thickness increases from some finite and small value toward infinity. In the latter case, surface tension effects are appreciable and have to be taken into account in the analysis of the problem.

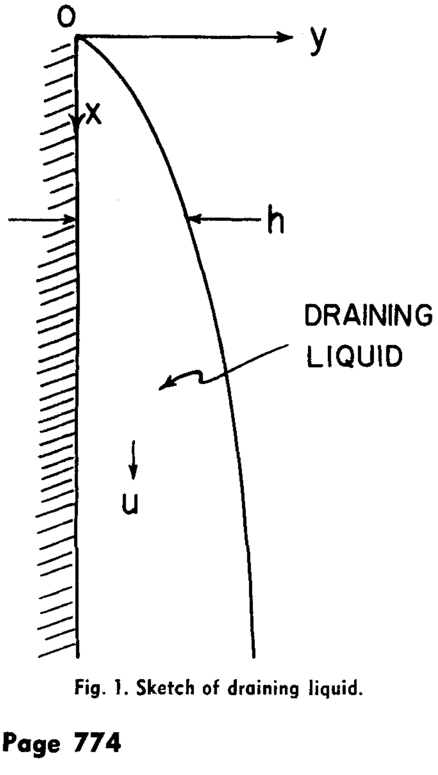

Fig. 1. Sketch of draining liquid.

In the case where the lower edge of the plate is not immersed, the change in curvature is not too large and surface tension can generally be neglected as compared to gravitational and viscous forces. In this communication the drainage of a fluid initially at rest from unimmersed bodies is considered and surface tension effects are neglected. The hydrodynamics near the lower edge of the body is not investigated.

In addition, it is assumed that the liquid film itself is stable and laminar without waves or ripples on the surace. The latter assumption holds well for the type of drainage described in this paper (*4*).

EFFECT OF ACCELERATION ON FLAT PLATE DRAINAGE

The first to investigate the drainage problem was Jeffreys (*3*). He considered the case of parallel flow and neglected inertia forces as compared with the viscous and gravitational ones. Thus, he wrote for the system described in Figure 1

$$0 = \nu \frac{\partial^2 u}{\partial y^2} + g \qquad (1)$$

with boundary conditions of zero velocity at the wall and zero shear rate at the liquid-air interface. A velocity profile, $u = f(y)$ was calculated for a fixed x. This profile was combined with the following continuity equation:

$$-\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(\int_0^h u\,dy\right) \qquad (2)$$

The result was an explicit expression for the thickness profile as a function of position, x, and time, t:

$$h = \sqrt{\frac{\nu x}{g t}} \qquad (3)$$

The instantaneous flow rate, per unit width of plate, past any point was:

$$q = \int_0^h u\,dy = \frac{g h^3}{3\nu} \qquad (4)$$

Equations (3) and (4) are based on the flow equation for the steady state [Equation (1)], and therefore they describe the hydrodynamics of the system only at the later stages of the drainage process.

A modification of this approach was presented by Green (*2*) and later extended by Wyllie (*6*). They started with the flow equation which included the acceleration term:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} + g \qquad (5)$$

The boundary conditions used were:

$$\begin{aligned} u\,(y, 0) &= 0 \\ \frac{\partial u}{\partial y}\,(h, t) &= 0 \\ u\,(0, t) &= 0 \end{aligned} \qquad (6)$$

Equation (5) was integrated from $y = 0$ to $y = h$ at constant x to yield the velocity profile in the film

$$u(y,t) = \frac{g}{\nu}\left[hy - \frac{y^2}{2} - \frac{2h^2}{\pi^3} \sum_{n=1}^{\infty} \frac{e^{-m^2 \nu t} \sin my}{(n-1/2)^3}\right]$$

where

$$m = (n - 1/2)\frac{\pi}{h} \qquad (7)$$

Wyllie also reported a thickness profile which he developed for the upper part of the liquid film. However, his reported profile was expressed in terms of two constants that had to be evaluated experimentally. In addition, his profile, for large values of time, did not converge to Jeffreys solution—Equation (3). A different solution that does not suffer from these shortcomings is herewith presented.

Integration of Equation (7) over the film thickness gives the total flow rate:

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$$q = \int_0^h u dy = \frac{gh^3}{3\nu}\left[1 - \frac{6}{\pi^4} \sum_{n=1}^{\infty} \frac{e^{-m^2 \nu t}}{(n - 1/2)}\right] \quad (8)$$

When this is substituted into the equation of continuity, Equation (2), the following differential equation is obtained:

$$-\frac{\partial h}{\partial t} = \left\{ \frac{gh^2}{\nu}\left[1 - \frac{6}{\pi^4} \sum_{n=1}^{\infty} \frac{e^{-m^2 \nu t}}{(n - 1/2)^2}\right] + - \frac{4gt}{\pi^2} \sum_{n=1}^{\infty} \frac{e^{-m^2 t}}{(n - 1/2)^2} \right\} \frac{\partial h}{\partial x} \quad (9)$$

This differential equation can be simplified by making use of the following mathematical relationship:

$$-\left(\frac{\partial h}{\partial t}\right)_x \Big/ \left(\frac{\partial h}{\partial x}\right)_t = \left(\frac{\partial x}{\partial t}\right)_h \quad (10)$$

When Equations (9) and (10) are combined, integrated, and the boundary condition $h(0, t) = 0$ is used, the implicit relationship for the film thickness profile is obtained, as shown in Equation (11):

$$x = \frac{gh^2}{\nu} t + \frac{2gh^4}{\pi^6 \nu^2} \sum_{n=1}^{\infty} \left\{ \frac{\exp\left[-\left(n - \frac{1}{2}\right)^2 \frac{\pi^2}{h^2} \nu t\right]}{(n - 1/2)^6} \left[5 + 2\left(n - \frac{1}{2}\right)^2 \frac{\pi^2}{h^2} \nu t\right] \right\} \quad (11)$$

For large values of time, Equations (8) and (11) do converge to those derived by Jeffreys, Equations (4) and (3), respectively.

The question that may be asked now is under what conditions could the simpler Jeffreys solution be substituted for Equation (11) with sufficient accuracy, say 1% difference from the exact solution? In order to answer this question, Equation (11) will be put into a dimensionless form. Defining a dimensionless thickness, T

$$T = h\sqrt{\frac{gt}{\nu x}} \quad (12)$$

and a time constant, F:

$$F = \frac{\nu t}{h^2} \quad (13)$$

one obtains from Equation (11)

$$T^{-2} = 1 + \frac{2}{\pi^2 F} \sum_{n=1}^{\infty}$$

INFORMATION RETRIEVAL*

An analytical study of heat and mass transfer in multiparticle systems at low Reynolds numbers, Pfeffer, Robert, and John Happel, **A.I.Ch.E. Journal, 10,** No. 5, p. 605 (September, 1964).

Key Words: Energy Equation-1, Free Surface Model-1, Creeping Motion Equations-1, Perturbation Procedure-10, Numerical Solution-10, Temperature Profile-2, Nusselt or Sherwood Number-7, j Factor-7, Reynolds Number-6, Peclet Number-6, Spherical Angle-6, Fractional Void Volume-6, Prandtl or Schmidt Number-6, Surface Chemical Reaction-6, Packed or Fluidized Beds-8.

Abstract: The problem of predicting particle-to-fluid heat and mass transfer rates in multiparticle systems at low Reynolds numbers was studied analytically. The energy equation coupled with the free surface model was solved by expanding the fluid temperature distribution in even powers of the spherical angle. Solutions were obtained for Peclet numbers between 0.1 and 100.0 and fractional void volumes between 0.4 and 1.0. A plot of the results in terms of the j factor vs. the Reynolds number shows good agreement with published experimental mass and heat transfer data and also indicates that the j factor is not independent of the Schmidt or Prandtl number at low Reynolds number, low Peclet number flow.

The frequency response analysis of a wetted wall adiabatic humidifier, Bruley, Duane F., and J. W. Prados, **A.I.Ch.E. Journal, 10,** No. 5, p. 612 (September, 1964).

Key Words: Distributed Parameter-1, Partial Differential Equations-1, Air-2, Water-2, Temperature-7, Frequency Response-8, Simultaneous Heat and Mass Transfer-8, Adiabatic Humidification-8, Laminar Flow-8, Turbulent Flow-8, Heat Transfer-8, Mass Transfer-8, Wetted Wall Column-9, Process Dynamics-9, Transfer Function-9, Bode Plot-10, Mathematical Solution-10, Computer-10.

Abstract: The dynamic thermal behavior of a wetted wall adiabatic humidifier was investigated. Partial differential equations describing the temperature-time-distance relations for the simultaneous heat and mass transfer system with laminar and turbulent air flow were derived in the time domain and solved in the frequency domain. Experimental frequency-response tests were made and the results compared with the theoretical models in the form of Bode plots. Good agreement between the theoretical and experimental studies was obtained.

Forced convection boiling of potassium-mercury systems, Tang, Y. S., P. T. Ross, R. C. Nicholson, and C. R. Smith, **A.I.Ch.E. Journal, 10,** No. 5 p. 617 (September, 1964).

Key Words: Boiling-8, Heat Transfer-8, Forced Convection-9, Wall Superheat-6, Potassium-1, Mercury-1, Composition-6, Net Boiling-8, Surface Boiling-8, Void Fraction-7, Vertical Tube-10, Electrical Heater-10.

Abstract: Forced convection boiling heat transfer performance is evaluated in a vertical test section of a forced circulation boiling loop with potassium-mercury binary systems of two different compositions. These amalgams are used in a thermally regenerative liquid-metal cell. Both net boiling and surface boiling heat transfer rates are reported. Data are compared with those of pure components in the literature.

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* For details on the use of these Key Words and the A.I.Ch.E. Information Retrieval Program, see **Chem. Eng. Progr.,** Vol. 60, No. 8, p. 88 (August, 1964). A free copy of this article may be obtained by sending a post card, with the words "Key Word Article" and your name and address (please print) to Publications Department, A.I.Ch.E., 345 East 47 St., N.Y. N.Y., 10017. Price quotations for volume quantities on request.

$$\left\{ \frac{\exp\left[-\left(n-\frac{1}{2}\right)^2 \pi^2 F\right]}{(n-1/2)^6} \left[5+2\left(n-\frac{1}{2}\right)^2 \pi^2 F\right]\right\} \qquad (14)$$

and for Jeffreys' Equation (3)

$$T = 1 \qquad (15)$$

If it is assumed that a difference of 1% in thickness, or T, is acceptable, one can find under what conditions Equation (15) may be used instead of Equation (14). If $T = 0.99$ for this case, one obtains the critical time constant from Equation (14):

$$F_{\text{crit}} = 3.80 \qquad (16)$$

or

$$h^2 = \nu t/3.80 \qquad (17)$$

When h is substituted from Equation (3) into Equation (17) one arrives at

$$t_{\text{crit}} = 1.95\sqrt{\frac{x}{g}} \qquad (18)$$

Thus, for a distance 100 cm. below the top of the plate the critical time is less than 0.7 sec. Moreover this time is independent of the fluid properties. The result obtained above indicates that in most practical cases the simple Jeffreys equation may be substituted for the more rigorous one developed here.

THE CYLINDER CASE

Another case where the applicability of the simple Jeffreys equation can be tested is the drainage down a cylindrical surface. Van Rossum (5) has reported that surface tension effects due to the curvature of the cylindrical surface are negligible for large cylinder radii. In dimensionless form, this criterion may be expressed as

$$N_{Go} = \text{Goucher number} > 1.8 \quad (19)$$

where

$$N_{Go} = R_w\sqrt{\frac{\rho g}{2\sigma}}$$

and

$$\sqrt{2\sigma/\rho g} = \text{capillary length.}$$

This dimensionless group, whose significance for cylinders was first reported in 1922 (*1*), may be considered as the ratio of gravitational force to surface tension. As the capillary length for most fluids is about 2 to 4 mm., Equation (19) indicates that, for cylinders with radii of several mm., surface tension, due to the curved cylindrical surface, is negligible. Therefore, drainage from many burettes, pipetes and large cylindrical surfaces may be described as those with high Goucher numbers, according to Equation (19). The flow equation with negligible inertia for this case is

$$0 = g + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \qquad (20)$$

The boundary conditions used are

$$\begin{aligned} u &= 0 \text{ at } r = R_w \\ \frac{\partial u}{\partial r} &= 0 \text{ at } r = R \end{aligned} \qquad (21)$$

Equation (20) may be integrated from $r = R_w$ to $r = R$ at constant x to give the velocity profile

$$u = \frac{g}{2\nu}\left[R^2 \ln\frac{R}{R_w} - \frac{r^2 - R_w^2}{2}\right] \qquad (22)$$

When Equation (22) is integrated, the total flow rate is obtained:

$$Q = 2\pi\int_{R_w}^{R} urdr = \frac{\pi g}{2\nu}\left[R^4 \ln\frac{R}{R_w} - \frac{3}{4}R^4 + R^2R_w^2 - \frac{R_w^2}{4}\right] \qquad (23)$$

When this is combined with the continuity balance

$$-\frac{\partial Q}{\partial x} = 2\pi R\frac{\partial R}{\partial t} \qquad (24)$$

one obtains

$$-\left[2R^2 \ln\frac{R}{R_w} - (R^2 - R_w^2)\right]\frac{\partial R}{\partial x} = \frac{2\nu}{g}\frac{\partial R}{\partial t} \qquad (25)$$

This equation is integrated with the boundary condition $R(0, t) = R_w$ to yield the relation between film thickness, x, and t:

$$x = \frac{gt}{\nu}\left[R^2 \ln\frac{R}{R_w} - \frac{1}{2}(R^2 - R_w^2)\right] \qquad (26)$$

When $h = R - R_w$ is substituted for drainage on a convex surface or $h = R_w - R$ for drainage on a concave surface, the following two expressions are obtained, respectively:

$$x = \frac{gt}{\nu}\left[(R_w + h)^2 \ln\left(1 + \frac{h}{R_w}\right) - h\left(R_w + \frac{h}{2}\right)\right] \qquad (27)$$

$$x = \frac{gt}{\nu}\left[(R_w - h)^2 \ln\left(1 - \frac{h}{R_w}\right) + h\left(R_w - \frac{h}{2}\right)\right] \qquad (28)$$

These equations may be rewritten in a dimensionless form by substituting Equation (12)

$$\left(\frac{\eta}{T}\right)^2 = (1 + \eta)^2 \ln(1 + \eta) - \eta\left(1 + \frac{\eta}{2}\right) \qquad (29)$$

INFORMATION RETRIEVAL

(Continued from page 778)

Virial coefficients for argon, methane, nitrogen, and xenon, Gyorog, Donald A., and Edward F. Obert, **A.I.Ch.E. Journal, 10,** No. 5, p. 621 (September, 1964).

Key Words: A. Virial Coefficients-2, B. Argon-8, Methane-8, Nitrogen-8, Xenon-8, C. Compressibility-1, Internal Energy Deviation-1, Enthalpy Deviation-1, Joule-Thomson Coefficient-1, D. Digital Computer-10, Curve Fitting-10, Linear-approximation Method-10, Least-Square Approximation Method-10, Least-Square Power Series-10, Orthogonal Polynominal Method-10.

Abstract: Several methods for deriving the virial coefficients from experimental property data are outlined. With these methods the second, third, and fourth virial coefficients for argon, methane, nitrogen, and xenon were established to satisfy experimental pvT, Δu^0, Δh^0, and Joule-Thomson coefficient data. The virial coefficients are tabulated for the following temperature ranges: nitrogen, 80° to 600°K.; argon, 120° to 600°K.; methane, 200° to 500°K.; and xenon, 270° to 450°K.

A generalized virial equation of state derived from experimental data, Gyorog, Donald A., and Edward F. Obert, **A.I.Ch.E. Journal, 10,** No. 5, p. 625 (September, 1964).

Key Words: Generalized Equation of State-2, Generalized Virial Coefficients and Derivatives-2, Corresponding States-8, Lennard-Jones Potential-9, Beattie-Bridgeman Equation-9, Benedict-Webb-Rubin Equation-9. B. Properties of Gases-9, Boyle Temperatures-10, Compressibility-9, Enthalpy Deviation-9, Internal Energy Deviation-9, Joule-Thomson Coefficients-9, Sonic Velocity-9. C. Argon-1, Nitrogen-1, Methane-1, Xenon-1, Carbon Monoxide-2, Oxygen-2, Air-2, Krypton-2, Neon-2. D. Digital Computer-10.

Abstract: Generalized B*, C*, D*, and their derivatives were established to satisfy thermodynamic properties for symmetrical, nonpolar gases. Accuracy and simplicity was demonstrated by property calculations and comparisons with other equations of state. Several methods for deriving the reduction parameters (ϵ/k and b_0) for the de Boer principle of corresponding states are outlined. Best correlation is obtained with reduction parameters derived from compressibility, but those derived from critical point properties are satisfactory. General agreement with theoretical coefficients calculated with the Lennard-Jones potential was found with the exception of C* and D* at low temperature.

Velocity distributions in two-dimensional laminar liquid-into-liquid jets in power-law fluids, Gutfinger, Chaim, and Reuel Shinnar, **A.I.Ch.E. Journal, 10,** No. 5, p. 631 (September, 1964).

Key Words: Fluid Flow-8, Jets-8, Flow-8, Laminar-, Two-Dimensional Fluids-9, Non-Newtonian-, Power-Law-, Mixing-9, Velocity-9, Carboxy-Methyl-Cellulose-1, Glycerine-1, Velocity Distribution-7, Velocity Profiles-7, Theory-10, Streak Photography-10.

Abstract: This paper describes an analytical and experimental investigation of a two-dimensional laminar liquid-into-liquid jet of a power-law fluid. The validity of boundary-layer assumptions is evaluated for this case and compared with the Newtonian one. It is shown that the range where these assumptions hold is narrower than in the Newtonian case. The applicability of the solution to pseudoplastic fluids with slightly elastic properties is discussed. Actual velocity distributions obtained for jets in dilute solutions of carboxy-methyl-cellulose are in reasonable agreement with those predicted by theory.

Gas phase controlled mass transfer in two-phase annular horizontal flow, Anderson, J. D., R. E. Bollinger, and D. E. Lamb, **A.I.Ch.E. Journal, 10,** No. 5, p. 640 (September, 1964).

Key Words: Ammonia-1, Annular-2, Contactor-3, Entrainment-4, Fluid Mechanics-5, Mass Transfer-6, 8, Pipe Line-7, Two Phase-9.

Abstract: Rates of mass transfer of ammonia from air to water have been measured in annular, concurrent, two-phase flow in a horizontal, 1-in. tube. Mass transfer coefficients ranged from 2.4 to 32 moles/(hr.) (sq. ft.) (atm.) for water flow rates between 190 and 2,000 lb./hr. and air rates between 80 and 430 lb./hr. The flow rates correspond to superficial liquid Reynolds numbers between 1,140 and 12,100 and superficial gas Reynolds numbers between 28,600 and 150,000. Entrainment of liquid in the gas core interchange of liquid between the core and the annulus and j_D factors for mass transfer were calculated from the concentration measurements. A method for predicting approximate gas-phase mass transfer coefficients is presented.

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$$\left(\frac{\eta}{T}\right)^2 = (1-\eta)^2 \ln (1-\eta) + \eta\left(1-\frac{\eta}{2}\right) \qquad (30)$$

where

$$\eta = \frac{h}{R_w}.$$

Once again under what conditions the single Jeffreys equation may be substituted for this case will be computed. Thus for a difference of 1% between the plate solution and the cylinder solution one obtains

$$\eta_{\text{crit}} = \left(\frac{h}{R_w}\right)_{\text{crit}} = 0.1 \qquad (31)$$

Thus when $h < 0.1\ R_w$, Jeffreys solution is a good approximation. This means that in many practical cases the simple Jeffreys equation may be used satisfactorily for drainage on a cylindrical surface.

ACKNOWLEDGMENT

This work was supported by National Science Foundation Grant G-19820.

NOTATION

F = time constant, $\nu t/h^2$, dimensionless
g = gravitational constant, LT^{-2}
h = thickness of the liquid film, L
m, n = summation indices
N_{Go} = Goucher number, $R_w\sqrt{\rho g/2\sigma}$, dimensionless
q = flow rate per unit width [defined by Equation (4)], L^2T^{-1}
Q = flow rate [defined by Equation (23)], L^3T^{-1}
r = radius, L
R = radius of curvature of the free surface of the liquid, L
R_w = wall, cylinder, or wire radius, L
T = dimensionless film thickness, $h\sqrt{gt/\nu x}$
t = time, T
u = velocity in x direction, LT^{-1}
x, y = coordinates, L

Greek Letters

η = ratio of thickness to wall radius, h/R_w, dimensionless
ν = kinematic viscosity, L^2T^{-1}
σ = surface tension, MT^{-2}
π = 3.1416

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